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Fermi Gas in *D*-Dimensional Space

Sami M. Al-Jaber[1]

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We consider a Fermi gas in D-dimensional space and show how the physical properties of the system behave as a function of the dimension D, in particular, the density of states, the Fermi energy, and the radius of the Fermi hypersphere.

1. INTRODUCTION

Recently, there has been much interest in the investigation of the dependence of physical systems upon the dimension D of space. It is believed that the dimension of space plays an important role in quantum field theory [1], in the Ising limit of quantum field theory [2], in random walks [3], and in the Casimir effect [4]. Other workers have discussed path integration of a relativistic particle in D-dimensional space [5, 6], and some authors have considered the relationship between the eigenstates of a hydrogen atom and a harmonic oscillator of arbitrary dimension [7, 8] and the construction of coherent states defined in a finite-dimensional Hilbert space [9–13].

The purpose of this paper is to consider the Fermi gas in D-dimensional space. The organization of this paper is as follows. In Section 2, I calculate the density of states $g(E)$, which is the number of particle quantum states per unit energy range; in Section 3, I derive the Fermi energy; Section 4 is a conclusion.

2. DENSITY OF STATES

Consider a large number of identical noninteracting spin-1/2 particles contained in a box with impenetrable walls each of length L. The time-independent Schrödinger equation inside the box is

[1] Department of Physics, An-Najah N. University, Nablus, West Bank, Via Israel; e-mail: Jaber@Najah.edu.

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$$\frac{-\hbar^2}{2m}\left[\sum_{i=1}^{D}\frac{\partial^2}{\partial x_i^2}\right]\psi(x_1, x_2, \ldots, x_D) = E\psi(x_1, x_2, \ldots, x_D) \tag{1}$$

The requirement that ψ vanishes at each of the walls gives the spatial part of the wave function as

$$\psi = C\prod_{i=1}^{D}\sin\left(\frac{n_i\pi X_i}{L}\right) \tag{2}$$

where the normalization constant is $C = (2/L)^{D/2}$. The corresponding allowed values of the energy of a particle are

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2 \tag{3}$$

where $n^2 = \Sigma_{i=1}^{D} n_i^2$. In order to calculate the density of states, it is useful to study the problem of the Fermi gas by imposing periodic boundary conditions on the spatial part of the wave functions of the particles, that is, by requiring these wave functions to be periodic in x_i with period L. We then have traveling plane-wave solutions of the form

$$\psi_k(\mathbf{r}) = \prod_{j=1}^{D}\exp[ik_jx_j] \tag{4}$$

where the allowed values of the components of the wave vector $\mathbf{k}$ are given by

$$k_j = \frac{2\pi}{L}n_j, \qquad j = 1, 2, \ldots, D \tag{5}$$

Because the particles have spin 1/2, each spatial orbital has S possible spin states. As was shown by Menon and Agrawal [14], S must be $D - 1$ for $D > 1$. Let N_s be the number of individual particle states having energies up to $E = \hbar^2k^2/2m$. These states are contained within a hypersphere in $\mathbf{k}$ space of radius k. Therefore,

$$N_s = (D-1)\left(\frac{L}{2\pi}\right)^D \times V_D \tag{6}$$

where V_D is the volume of the hypersphere, which in D-dimensional hyperspherical coordinates is

$$V_D = \int r^{D-1}dr\, d\Omega_D \tag{7}$$

where the differential solid angle is

$$d\Omega_D = d\theta_1 \sin\theta_2\, d\theta_2 \sin^2\theta_3\, d\theta_3 \cdots \sin^{D-2}\theta_{D-1}\, d\theta_{D-1}$$

and the domain of integration is

$$r:\quad 0 \to \mathrm{k},\qquad \theta_1:\quad 0 \to 2\pi,\qquad \theta_i:\quad 0 \to \pi \qquad (2 \le i \le D-1)$$

The integration in Eq. (7) is straightforward and the result is [15]

$$V_D = \frac{\pi^{D/2} K^D}{\Gamma(1 + D/2)} \tag{8}$$

where $\Gamma(x)$ is the gamma function. Therefore Eq. (6) becomes

$$N_S = (D-1)\left(\frac{L}{2\pi}\right)^D \frac{\pi^{D/2}}{\Gamma(1 + D)} K^D \tag{9}$$

Using $E = \hbar^2 k^2/2m$, we can rewrite Eq. (9)

$$N_S = (D-1)\left(\frac{L}{2\pi}\right)^D \frac{\pi^{D/2}}{\Gamma(1 + D/2)} \left(\frac{2m}{\hbar^2}\right)^{D/2} E^{D/2} \tag{10}$$

The density of states, $g(E)$ is defined as the number of particle quantum states per unit energy range. The number $g(E)\, dE$ of particles states within the energy range $(E, E + dE)$ is thus given by dN_S, so that

$$g(E) = \frac{dN_S}{dE} = \frac{D}{2}(D-1)\left(\frac{L}{2\pi}\right)^D \frac{\pi^{D/2}}{\Gamma(1 + D/2)} \left(\frac{2m}{\hbar^2}\right)^{D/2} E^{(D-2)/2} \tag{11}$$

3. THE FERMI ENERGY

The Fermi energy E_F can be evaluated by requiring that the total number N of particles in the system should be equal to

$$\begin{aligned} N &= \int_0^{E_F} g(E)\, dE \\ &= (D-1)\left(\frac{L}{2\pi}\right)^D \frac{\pi^{D/2}}{\Gamma(1 + D/2)} \left(\frac{2m}{\hbar^2}\right)^{D/2} E_F^{D/2} \end{aligned} \tag{12}$$

so that

$$E_F = \frac{\hbar^2}{2m} 4\pi \left[\frac{\Gamma(1 + D/2)}{D-1}\rho\right]^{2/D} \tag{13}$$

where

$$\rho = \frac{N}{V} \tag{14}$$

is the number of particles per unit volume, and $V = L^D$. Our result for E_F yields the well-known expression in three-dimensional space which is found in most quantum mechanics textbooks [16],

$$E_F\,(D = 3) = \frac{\hbar^2}{2m}(3\pi^2\rho)^{2/3} \tag{15}$$

The total energy of a Fermi gas in the ground state (at absolute temperature $T = 0$) is

$$\begin{aligned} E_{tot} &= \int_0^{E_F} Eg(E)\, dE \\ &= \frac{D(D-1)V}{(D+2)\,(2\pi)^D}\left(\frac{2m}{\hbar^2}\right)^{D/2}\frac{\pi^{D/2}}{\Gamma(1+D/2)}\,E_F^{(D+2)/2} \end{aligned} \tag{16}$$

and using Eq. (12), we get

$$E_{tot} = \frac{D}{D+2}\,NE_F \tag{17}$$

The average particle energy at $T = 0$ is therefore

$$\bar{E} = \frac{E_{tot}}{N} = \frac{D}{D+2}\,E_F \tag{18}$$

It is clear that for the three-dimensional case ($D = 3$) Eq. (18) gives the well-known result $\bar{E} = \frac{3}{5}E_F$.

All occupied single-particles states of a Fermi gas at zero absolute temperature fill a hypersphere in **k** space, having a radius K_F. This is called the Fermi hypersphere, and Eq. (12) gives

$$N = (D-1)\frac{V}{(4\pi)^{D/2}}\frac{K_F^D}{\Gamma(1+D/2)}$$

which implies

$$K_F = \left[\frac{(4\pi)^{D/2}}{D-1}\Gamma\left(1+\frac{D}{2}\right)\rho\right]^{1/D} \tag{19}$$

Again the above result yields the three-dimensional result for the radius of the Fermi sphere, namely

$$K_F = (3\pi^2\rho)^{1/3} \qquad \text{for} \quad D = 3$$

4. CONCLUSION

In this paper we have shown the dependence of a physical system upon the dimesion D of space. In particular, we have considered a Fermi gas in D dimensions and derived the density of states, the Fermi energy, and the radius of the Fermi hypersphere. All these results were shown to depend on the dimension D, and yield the expected results in three-dimensional space.

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